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Quantum and classical eigenfunctions in Calogero and Sutherland systems

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Abstract

An interesting observation was reported by Corrigan–Sasaki that all the frequencies of small oscillations around equilibrium are ‘quantized’ for Calogero and Sutherland (CS) systems, typical integrable multi-particle dynamics. We present an analytic proof by applying recent results of Loris–Sasaki. Explicit forms of ‘classical’ and quantum eigenfunctions are presented for CS systems based on any root system.

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1. Introduction

In a recent paper [1] simple theorems pertaining to the correspondence between quantum and classical dynamics are proved for the general multi-particle quantum mechanical systems with discrete eigenvalues. The theorems relate quantum mechanical eigenvalues and eigenfunctions to the properties of the classical mechanical system at equilibrium. Corresponding to each quantum eigenfunction, a ‘classical eigenfunction’ is defined whose eigenvalue is given by the ‘main part’, that is the order \hbar part, of the quantum eigenvalue. For the ‘elementary excitations’ these classical and quantum eigenvalues are nothing but the eigenfrequencies of the normal modes of the small oscillations at equilibrium.

We apply these theorems to the Calogero and Sutherland [2] systems, typical integrable multi-particle dynamics with long range interactions based on root systems [3]. The theorems provide an analytic proof for the interesting observations made by Corrigan–Sasaki [4–6] concerning the classical and quantum integrability in Calogero and Sutherland systems. Explicit forms of the classical and quantum eigenfunctions for the elementary excitations are presented for the Calogero and Sutherland systems based on any root systems. These exemplify another aspect of the close relationship between the classical and quantum integrability in Calogero and Sutherland systems.

This paper is organized as follows. In section 2, a basic formulation of multi-particle quantum mechanics in terms of the *prepotential* [7, 8] is briefly reviewed. After the

reformulation of the quantum mechanical wavefunctions at equilibrium, the main theorem of the Loris–Sasaki paper [1] is recapitulated. In section 3 the basic concepts of the Calogero and Sutherland systems are summarized. Sections 4 and 5 are the main part of this paper, presenting the classical and quantum eigenfunctions of the Calogero systems (section 4) and Sutherland systems (section 5). The final section is for summary and comments.

2. Basic quantum mechanics

Let us start with a basic formulation of a multi-particle quantum mechanical system in terms of a *prepotential* [7, 8] and later we will discuss its relationship with the corresponding classical ($\hbar \rightarrow 0$) dynamics. The dynamical variables are the coordinates $\{q_j | j = 1, \dots, r\}$ and their canonically conjugate momenta $\{p_j | j = 1, \dots, r\}$, subject to the Heisenberg commutation relations or the Poisson bracket relations:

$$\begin{aligned} [q_j, p_k] &= i\hbar\delta_{jk} & [q_j, q_k] &= [p_j, p_k] = 0 \\ \{q_j, p_k\} &= \delta_{jk} & \{q_j, q_k\} &= \{p_j, p_k\} = 0. \end{aligned}$$

We will adopt the standard vector notation in \mathbf{R}^r ,

$$q = (q_1, \dots, q_r) \quad p = (p_1, \dots, p_r) \quad q^2 \equiv \sum_{j=1}^r q_j^2 \quad p^2 \equiv \sum_{j=1}^r p_j^2, \dots \quad (2.1)$$

in which r is the number of particles. In quantum theory, the momentum operator p_j acts as a differential operator:

$$p_j = -i\hbar \frac{\partial}{\partial q_j} \quad j = 1, \dots, r.$$

Throughout this paper we discuss the standard Hamiltonian system

$$H = \frac{1}{2}p^2 + V(q) \quad (2.2)$$

in which we have assumed for simplicity that all the particles have the same mass, which is rescaled to unity. Let us start with the mild assumptions that the system has a unique and *square integrable* ground state ψ_0 ,

$$H\psi_0 = 0 \quad \int |\psi_0|^2 d^r q < \infty \quad (2.3)$$

and that it has a finite (or an infinite) number of *discrete* eigenvalues:

$$H\psi_n = E_n\psi_n \quad E_n = \varepsilon_n\hbar + \mathcal{O}(\hbar^2). \quad (2.4)$$

Here we adopt the convention that the ground state energy is vanishing, by adjusting the constant part of the potential V , see below.

Since the above time-independent Schrödinger equation is real for a self-adjoint Hamiltonian and the ground state has no *node* we express the ground state eigenfunction as

$$\psi_0(q) = \exp\left(\frac{1}{\hbar}W(q)\right) \quad (2.5)$$

in which a real function $W = W(q)$ is called a *prepotential* [7, 8]. By simple differentiation of (2.5), we obtain

$$p_j\psi_0 = -i\frac{\partial W}{\partial q_j}\psi_0 \quad p^2\psi_0 = -\sum_{j=1}^r \left[\left(\frac{\partial W}{\partial q_j}\right)^2 + \hbar\frac{\partial^2 W}{\partial q_j^2} \right] \psi_0 \quad (2.6)$$

which results in

$$\left\{ \frac{1}{2}p^2 + \frac{1}{2} \sum_{j=1}^r \left[\left(\frac{\partial W}{\partial q_j} \right)^2 + \hbar \frac{\partial^2 W}{\partial q_j^2} \right] \right\} \psi_0 = 0. \quad (2.7)$$

In other words, we can express the potential (plus the ground state energy) in terms of the prepotential [4, 7, 8]³

$$V(q) = \frac{1}{2} \sum_{j=1}^r \left[\left(\frac{\partial W}{\partial q_j} \right)^2 + \hbar \frac{\partial^2 W}{\partial q_j^2} \right]. \quad (2.8)$$

By removing the obvious \hbar -dependent terms, let us define a *classical* potential $V_C(q)$:

$$V_C(q) = \frac{1}{2} \sum_{j=1}^r \left(\frac{\partial W}{\partial q_j} \right)^2. \quad (2.9)$$

Conversely, (2.8) is a Riccati equation determining the prepotential W for a given potential V (or V_C). Needless to say, it does not matter if the prepotential can be expressed in terms of elementary functions or not.

2.1. Equilibrium position and frequencies of small oscillations

Now let us consider the equilibrium point of the classical potential V_C (2.9). The classical Hamiltonian $H_C = p^2/2 + V_C$ has a stationary solution at the classical equilibrium point, $p = 0, q = \bar{q}$. There could be, in general, many stationary points of the classical potential V_C , among which we will focus on the ‘*maximum*’ point \bar{q} of the ground state wavefunction ψ_0 [4]:

$$\left. \frac{\partial W}{\partial q_j} \right|_{\bar{q}} = 0 \implies \left. \frac{\partial V_C}{\partial q_j} \right|_{\bar{q}} = \sum_{k=1}^r \left. \frac{\partial^2 W}{\partial q_j \partial q_k} \right|_{\bar{q}} \left. \frac{\partial W}{\partial q_k} \right|_{\bar{q}} = 0 \quad j = 1, \dots, r. \quad (2.10)$$

By expanding the classical potential V_C around \bar{q} (2.10), we obtain

$$\begin{aligned} V_C(q) &= \frac{1}{2} \sum_{j,k=1}^r \left. \frac{\partial^2 V_C}{\partial q_j \partial q_k} \right|_{\bar{q}} (q - \bar{q})_j (q - \bar{q})_k + \mathcal{O}((q - \bar{q})^3) \\ &= \frac{1}{2} \sum_{j,k,l=1}^r \left. \frac{\partial^2 W}{\partial q_j \partial q_l} \right|_{\bar{q}} \left. \frac{\partial^2 W}{\partial q_l \partial q_k} \right|_{\bar{q}} (q - \bar{q})_j (q - \bar{q})_k + \mathcal{O}((q - \bar{q})^3) \end{aligned} \quad (2.11)$$

since $V_C(\bar{q}) = 0$, (2.9). Thus the eigen(angular)frequencies ((frequency)²) of small oscillations near the classical equilibrium are given as the eigenvalues of the Hessian matrix \tilde{W} (\tilde{V}_C):

$$\tilde{W} = \text{Matrix} \left[\left. \frac{\partial^2 W}{\partial q_j \partial q_k} \right|_{\bar{q}} \right] \quad \tilde{V}_C = \text{Matrix} \left[\left. \frac{\partial^2 V_C}{\partial q_j \partial q_k} \right|_{\bar{q}} \right] = \tilde{W}^2. \quad (2.12)$$

³ Similar formulae can be found within the context of supersymmetric quantum mechanics [9]. Here we stress that supersymmetry is not necessary.

2.2. Classical limit of quantum eigenfunctions

Let us express the discrete eigenfunctions in product forms

$$\psi_n(q) = \phi_n(q)\psi_0(q) \quad n = 0, 1, \dots \quad \phi_0 \equiv 1 \quad (2.13)$$

in which ϕ_n obeys a simplified equation with the similarity transformed Hamiltonian \hat{H} [7, 8]:

$$\hat{H}\phi_n = E_n\phi_n \quad (2.14)$$

$$\hat{H} = \exp\left(-\frac{1}{\hbar}W\right) H \exp\left(\frac{1}{\hbar}W\right) = -\frac{\hbar^2}{2}\Delta + \hbar\hat{A} \quad (2.15)$$

in which Δ is the Laplacian and a linear differential operator \hat{A} is defined for any smooth function $\varphi(q)$ as

$$(\hat{A}\varphi)(q) \equiv -\sum_{j=1}^r \frac{\partial W(q)}{\partial q_j} \frac{\partial \varphi(q)}{\partial q_j} \quad \Delta = \sum_{j=1}^r \frac{\partial^2}{\partial q_j^2}. \quad (2.16)$$

Here we adjust the normalization of the eigenfunctions $\{\phi_n\}$ so that the corresponding ‘classical’ eigenfunctions $\{\varphi_n\}$ are finite (non-vanishing) in the limit $\hbar \rightarrow 0$:

$$\lim_{\hbar \rightarrow 0} \phi_n(q) = \varphi_n(q) \quad n = 1, 2, \dots \quad (2.17)$$

By taking the classical limit ($\hbar \rightarrow 0$) of (2.14) and considering (2.4), (2.15), we arrive at an ‘eigenvalue equation’ for the ‘classical’ wavefunctions

$$\hat{A}\varphi_n = \mathcal{E}_n\varphi_n \quad n = 1, 2, \dots \quad (2.18)$$

in which the operator \hat{A} is defined above (2.16). Conversely one could start with the above eigenvalue equation. One defines the classical eigenfunctions as its solutions satisfying certain regularity conditions. Then the *quantum* eigenfunction ϕ_n could be considered as an \hbar -deformation of the *classical* eigenfunction φ_n . For the Calogero and Sutherland systems to be discussed below, there is a one-to-one correspondence between the classical and quantum eigenfunctions. For generic multi-particle quantum mechanical systems, the situation is less clear.

2.3. Theorems

The *classical* eigenfunctions have the following remarkable properties:

Proposition 2.1. *The product of two classical eigenfunctions $(\varphi_n, \mathcal{E}_n)$ and $(\varphi_m, \mathcal{E}_m)$ is again a classical eigenfunction with the eigenvalue $\mathcal{E}_n + \mathcal{E}_m$,*

$$-\sum_{j=1}^r \frac{\partial W}{\partial q_j} \frac{\partial(\varphi_n\varphi_m)}{\partial q_j} = (\mathcal{E}_n + \mathcal{E}_m)\varphi_n\varphi_m. \quad (2.19)$$

Proposition 2.2. *The classical eigenfunctions vanish at the equilibrium \bar{q}*

$$\varphi_n(\bar{q}) = 0 \quad n = 1, 2, \dots \quad (2.20)$$

Proposition 2.3. *The derivatives of a classical eigenfunction at the equilibrium \bar{q} form an eigenvector of the Hessian matrix \tilde{W} , iff $\nabla\varphi_n|_{\bar{q}} \neq 0$*

$$-\tilde{W} \cdot \nabla\varphi_n|_{\bar{q}} = \mathcal{E}_n \nabla\varphi_n|_{\bar{q}} \quad n = 1, 2, \dots \quad (2.21)$$

or

$$-\sum_{j=1}^r \frac{\partial^2 W}{\partial q_k \partial q_j} \Big|_{\bar{q}} \frac{\partial \varphi_n}{\partial q_j} \Big|_{\bar{q}} = \mathcal{E}_n \frac{\partial \varphi_n}{\partial q_k} \Big|_{\bar{q}} \quad n = 1, 2, \dots \tag{2.22}$$

Obviously the Hessian matrix \tilde{W} has at most r different eigenvalues and eigenvectors. The classical eigenfunctions $\{(\varphi_j, \mathcal{E}_j)\}$, $j = 1, \dots, r$ for which $\nabla \varphi_j|_{\bar{q}} \neq 0$ will be called ‘*elementary excitations*’. At equilibrium, each corresponds to the *normal coordinate* of the small oscillations with the eigen (angular) frequency \mathcal{E}_j . The elementary excitations are the generators of all the classical eigenfunctions. In other words, any classical eigenfunction can be expressed as

$$\varphi_1^{n_1} \cdots \varphi_r^{n_r} \quad \mathcal{E} = n_1 \mathcal{E}_1 + \cdots + n_r \mathcal{E}_r \quad n_j \in \mathbb{Z}_+ \tag{2.23}$$

or a linear combination thereof with the same eigenvalue \mathcal{E} . The above type of classical eigenfunctions are obviously non-elementary and they have zero gradient at equilibrium, for example, $\nabla(\varphi_j \varphi_k)|_{\bar{q}} = 0$. Because of this property, the representation of the elementary excitations is not unique except for some lower members.

These results provide a basis of the analytical proof of the observations made in the Corrigan–Sasaki paper [4] on the correspondence/contrast between the *classical* and *quantum* integrability in Calogero–Moser systems. It should be mentioned that Perelomov’s recent work [6] asserts essentially our proposition 2.3 for the special cases of the quantum–classical eigenvalue correspondence of the Sutherland systems.

Throughout this section we have assumed that the prepotential W is independent of the Planck’s constant \hbar , for simplicity of the presentation. The main content of this section is valid even if W depends on \hbar , so long as $\lim_{\hbar \rightarrow 0} W = W_0$ is well defined. A celebrated example that $\lim_{\hbar \rightarrow 0} W$ diverges is the hydrogen atom, for which the classical equilibrium does not exist. In this case the quantum–classical correspondence does not make sense and the present formulation does not apply.

In the subsequent sections, we will show many explicit examples of the *classical* and *quantum* eigenfunctions and their relationship.

3. Root systems and Calogero–Moser dynamics

A Calogero–Moser system is a multi-particle Hamiltonian dynamics associated with a root system Δ of rank r . This is a set of vectors in \mathbf{R}^r invariant under reflections in the hyperplane perpendicular to each vector in Δ :

$$\Delta \ni s_\alpha(\beta) = \beta - (\alpha^\vee \cdot \beta)\alpha \quad \alpha^\vee = \frac{2\alpha}{\alpha^2} \quad \alpha, \beta \in \Delta. \tag{3.1}$$

The set of reflections $\{s_\alpha, \alpha \in \Delta\}$ generates a finite reflection group G_Δ , known as a Coxeter (or Weyl) group.

A Calogero–Moser system is integrable both at the classical and quantum levels for various choices of the long range interaction potentials; rational ($1/q^2$), rational with a harmonic confining potential, trigonometric ($1/\sin^2 q$), hyperbolic ($1/\sinh^2 q$) and elliptic with the Weierstrass function ($\wp(q)$) potential. In the rest of this paper we will discuss the rational case (with a harmonic confining potential) under the name of Calogero systems [2] and the trigonometric potential case to be called Sutherland systems [2]. Both quantum Hamiltonians have an infinite number of discrete eigenvalues. The prepotentials are

$$\text{Calogero: } W = W_R - \frac{\omega}{2} q^2 \quad W_R = \sum_{\rho \in \Delta_+} g_\rho \log \rho \cdot q \tag{3.2}$$

in which W_R is the prepotential of the theory without the harmonic confining potential and

$$\text{Sutherland: } W = \sum_{\rho \in \Delta_+} g_\rho \log \sin(\rho \cdot q). \quad (3.3)$$

In these formulae, Δ_+ is the set of positive roots and g_ρ are real *positive* coupling constants which are defined on orbits of the corresponding Coxeter group, i.e. they are identical for roots in the same orbit. For crystallographic root systems there is one coupling constant $g_\rho = g$ for all roots in simply laced models, and there are two *independent* coupling constants, $g_\rho = g_L$ for long roots and $g_\rho = g_S$ for short roots in non-simply laced models. We will give the explicit forms of W in later sections. Throughout this paper we put the scale factor in the trigonometric functions to unity for simplicity; instead of the general form $a^2/\sin^2 a(\rho \cdot q)$, we use $1/\sin^2(\rho \cdot q)$. We also adopt the convention that long roots have squared length 2, $\rho_L^2 = 2$, unless otherwise stated. These prepotentials determine the potentials:

$$V = \begin{cases} \frac{\omega^2}{2} q^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} \frac{g_\rho (g_\rho - \hbar) \rho^2}{(\rho \cdot q)^2} - E_0 & \text{Calogero} \\ \frac{1}{2} \sum_{\rho \in \Delta_+} \frac{g_\rho (g_\rho - \hbar) \rho^2}{\sin^2(\rho \cdot q)} - E_0 & \text{Sutherland.} \end{cases} \quad (3.4)$$

The Sutherland systems are integrable, both at the classical and quantum levels, for the crystallographic root systems, that is those associated with simple Lie algebras: $\{A_r, r \geq 1\}$,⁴ $\{B_r, r \geq 2\}$, $\{C_r, r \geq 2\}$, $\{D_r, r \geq 4\}$, E_6 , E_7 , E_8 , F_4 and G_2 and the so-called $\{BC_r, r \geq 2\}$. On the other hand, the Calogero systems are integrable for any root system, crystallographic and non-crystallographic. The latter are H_3 , H_4 and $\{I_2(m), m \geq 4\}$, the dihedral group of order $2m$.

The prepotential W (3.2), (3.3), and hence the (classical) potential V (3.4) and the Hamiltonian are Coxeter (Weyl) invariant,

$$\begin{aligned} W(s_\rho(q)) &= W(q) & V(s_\rho(q)) &= V(q) & V_C(s_\rho(q)) &= V_C(q) & \forall \rho \in \Delta \\ H(s_\rho(p), s_\rho(q)) &= H(p, q) & \hat{H}(s_\rho(p), s_\rho(q)) &= \hat{H}(p, q) & \hat{A}(s_\rho(q)) &= \hat{A}(q) \end{aligned} \quad (3.5)$$

which is the symmetry of all the Calogero–Moser systems. This results in the fact that the ground state ψ_0 and all the other eigenfunctions are Coxeter (Weyl) invariant [8]:

$$\begin{aligned} \psi_0(s_\rho(q)) &= \psi_0(q) & \psi_n(s_\rho(q)) &= \psi_n(q) \\ \phi_n(s_\rho(q)) &= \phi_n(q) & \varphi_n(s_\rho(q)) &= \varphi_n(q). \end{aligned} \quad (3.6)$$

The quantum Calogero and Sutherland systems are not only integrable but also *exactly solvable* [8], that is, the similarity transformed Hamiltonians (2.15) are *lower triangular* in certain bases of the Hilbert space. The eigenvalues can be read easily from the diagonal matrix elements of \hat{H} . The exact eigenvalues of the excited states in the Calogero system are an integer multiple of the oscillator quantum $\omega\hbar$:

$$E_{\vec{n}} = \omega\hbar \sum_{j=1}^r n_j f_j \quad n_j \in \mathbb{Z}_+. \quad (3.7)$$

Here $\vec{n} = (n_1, \dots, n_r)$ are non-negative *quantum numbers* and $f_j = 1 + e_j$ (see table 1) and the integers $\{e_j\}$, $j = 1, \dots, r$ are called the *exponents* of the root system Δ .

⁴ For A_r models, it is customary to introduce one more degree of freedom, q_{r+1} and p_{r+1} and embed all of the roots in \mathbf{R}^{r+1} .

Table 1. The degrees f_i at which an elementary excitation exists.

Δ	$f_j = 1 + e_j$	Δ	$f_j = 1 + e_j$
A_r	$2, 3, 4, \dots, r + 1$	E_8	$2, 8, 12, 14, 18, 20, 24, 30$
B_r	$2, 4, 6, \dots, 2r$	F_4	$2, 6, 8, 12$
C_r	$2, 4, 6, \dots, 2r$	G_2	$2, 6$
D_r	$2, 4, \dots, 2r - 2; r$	$I_2(m)$	$2, m$
E_6	$2, 5, 6, 8, 9, 12$	H_3	$2, 6, 10$
E_7	$2, 6, 8, 10, 12, 14, 18$	H_4	$2, 12, 20, 30$

The coupling constant(s) g_ρ of the rational $1/q^2$ potentials shifts *only* the ground state energy E_0 in (3.4),

$$E_0 = \omega\hbar r/2 + \omega \sum_{\rho \in \Delta_+} g_\rho \tag{3.8}$$

in which the first term is the zero-point energy of the oscillators. For a given non-negative integer N , let $\mathcal{P}(N)$ be the number of different solutions of

$$N = \sum_{j=1}^r n_j f_j \quad n_j \in \mathbb{Z}_+. \tag{3.9}$$

Then the energy eigenvalue $E = \omega\hbar N + E_0$ has the degeneracy $\mathcal{P}(N)$.

The exact eigenvalues of the excited states in the Sutherland [8, 10] system are specified by the *dominant highest weight* $\lambda_{\vec{n}}$:

$$E_{\vec{n}} = 2\hbar^2 \lambda_{\vec{n}}^2 + 4\hbar \lambda_{\vec{n}} \cdot \varrho \tag{3.10}$$

$$\lambda_{\vec{n}} = \sum_{j=1}^r n_j \lambda_j \quad n_j \in \mathbb{Z}_+ \tag{3.11}$$

$$\varrho = \frac{1}{2} \sum_{\rho \in \Delta_+} g_\rho \rho \quad \delta = \frac{1}{2} \sum_{\rho \in \Delta_+} \rho. \tag{3.12}$$

Here $\vec{n} = (n_1, \dots, n_r)$ are non-negative *quantum numbers*, $\lambda_j, j = 1, \dots, r$ are the *fundamental weights* and δ and ϱ are called the *Weyl vector* and a *deformed Weyl vector*. The ground state energy E_0 in (3.4) is solely determined by ϱ :

$$E_0 = 2\varrho^2. \tag{3.13}$$

For the general discussion of quantum Calogero and Sutherland systems for any root system along the present line of arguments, the quantum integrability, Lax pairs, quantum eigenfunctions, creation–annihilation operators etc, we refer to [8]. A rather different approach by Heckman and Opdam [10] to Calogero–Moser models with degenerate potentials based on any root system should also be mentioned in this connection. The eigenfunctions of the Sutherland systems are sometimes called Heckman–Opdam’s Jacobi polynomials. Those for the *A-series* are known as the Jack polynomials [11].

In the following two sections, we will show the classical and quantum eigenfunctions of the elementary excitations in Calogero systems (section 4) and in Sutherland systems (section 5). For brevity and clarity of the presentation, we present the eigenfunctions of the ‘*reduced theory*’ in which most of the coupling constants are put to unity. To be more precise, for simply laced theories (*A, D, E, H* and I_2 (odd)) we put the coupling constant unity, $g = 1$. For non-simply laced theories (*B, C, F_4, G_2* and I_2 (even)) we put the coupling constant for

long roots unity $g_L = 1$ and keep the coupling constant for short roots intact, $g_S = \gamma$. The angular frequency of the harmonic confining potential is also put to unity, $\omega = 1$.

Let us introduce elementary symmetric polynomials as useful ingredients for expressing the eigenfunctions. The degree k elementary symmetric polynomial in r variables, $\{t_1, t_2, \dots, t_r\}$, $S_k(\{t_j\})$ is defined by the expansion of a generating function

$$G(x; \{t_j\}) = \prod_{j=1}^r (x + t_j) = \sum_{k=0}^r S_k(\{t_j\}) x^{r-k}. \quad (3.14)$$

4. Classical and quantum eigenfunctions of the Calogero systems

The basis of the quantum eigenfunctions $\{\phi_n\}$ of the Calogero system is the Coxeter (Weyl) invariant polynomials in the coordinates $\{q_j\}$. In order to express the eigenfunctions in a closed form, let us introduce the similarity transformed Hamiltonian \hat{H}_R *without the harmonic confining potential*,

$$\hat{H}_R = \hat{H} - \omega \hbar \mathcal{D} \quad \mathcal{D} = q \cdot \nabla = \sum_{j=1}^r q_j \frac{\partial}{\partial q_j} \quad (4.1)$$

$$\hat{H}_R = -\frac{\hbar^2}{2} \Delta + \hbar \hat{A}_R \quad \hat{A}_R = -\nabla W_R \cdot \nabla = -\sum_{j=1}^r \frac{\partial W_R}{\partial q_j} \frac{\partial}{\partial q_j} \quad (4.2)$$

in which \mathcal{D} is the Euler derivative measuring the degree of a monomial. The Hamiltonian \hat{H}_R maps a Coxeter invariant polynomial to another *with degree 2 less*

$$[\mathcal{D}, \hat{H}_R] = -2\hat{H}_R \quad (4.3)$$

which implies for an arbitrary parameter $\kappa \in \mathbb{C}$

$$(\mathcal{D} + 2\kappa \hat{H}_R) \exp[\kappa \hat{H}_R] = \exp[\kappa \hat{H}_R] \mathcal{D}.$$

The *lower triangularity* of the Hamiltonian \hat{H}_R means that the exponential operator contains only finite powers of \hat{H}_R , up to $[N/2]$, when applied to a Coxeter invariant polynomial of degree N . By multiplying $\omega \hbar$ on both sides and choosing $\kappa = 1/2\omega \hbar$, we obtain

$$\hat{H} \exp[\hat{H}_R/2\omega \hbar] = \exp[\hat{H}_R/2\omega \hbar] \omega \hbar \mathcal{D}. \quad (4.4)$$

Thus we arrive at a formula for an eigenfunction of \hat{H} with the eigenvalue $\omega \hbar N$ (N being a non-negative integer), starting from an arbitrary *homogeneous* Coxeter invariant polynomial $\Phi_N(q)$ of degree N :

$$\hat{H} \phi_N(q) = \omega \hbar N \phi_N(q) \quad (4.5)$$

$$\phi_N(q) \equiv \exp[\hat{H}_R/2\omega \hbar] \Phi_N(q) \quad \mathcal{D} \Phi_N(q) = N \Phi_N(q). \quad (4.6)$$

A similar formula was derived in [12] for the theories based on the A -series of root systems. There are $\mathcal{P}(N)$ (3.9) linearly independent Coxeter invariant homogeneous polynomials, which is equal to the degeneracy of the eigenspace of $E = \omega \hbar N$. Among them there are special eigenfunctions which are linear combinations of the Coxeter invariant *homogeneous* polynomials such that they are annihilated by \hat{H}_R :

$$\hat{H}_R \Phi_N = 0 \implies \hat{H} \Phi_N(q) = \omega \hbar N \Phi_N(q). \quad (4.7)$$

The number of homogeneous eigenfunctions is $\mathcal{P}(N) - \mathcal{P}(N - 2)$, which is much smaller than the total dimensionality of the eigenspace, $\mathcal{P}(N)$.

The simplest class of quantum eigenfunctions depends only on $q^2 = \sum_j q_j^2$, (2.1):

$$\begin{aligned}\hat{A}_R(q^2)^n &= -2n \sum_{\rho \in \Delta_+} g_\rho (q^2)^{n-1} & \Delta(q^2)^n &= 4n(r/2 + n - 1)(q^2)^{n-1} \\ \hat{H}_R(q^2)^n &= -2n\hbar \left(\frac{r}{2}\hbar + \sum g_\rho + \hbar(n-1) \right) (q^2)^{n-1} \\ \hat{H}_R \left(\frac{\omega}{\hbar} q^2 \right)^n &= -2n\omega\hbar \left(\frac{r}{2} + \frac{1}{\hbar} \sum g_\rho + (n-1) \right) \left(\frac{\omega}{\hbar} q^2 \right)^{n-1}.\end{aligned}\quad (4.8)$$

If we define $x \equiv \omega q^2 / \hbar$ and $\alpha \equiv r/2 + \sum_\rho g_\rho / \hbar - 1 = E_0 / \hbar \omega - 1$ (with E_0 defined in (3.8)), we obtain the associated Laguerre polynomial in x as the quantum eigenfunction

$$\exp \left[\frac{\hat{H}_R}{2\omega\hbar} \right] (-1)^n \frac{x^n}{n!} = \sum_{j=0}^n (-1)^j \binom{\alpha+n}{n-j} \frac{x^j}{j!} = L_n^{(\alpha)}(x). \quad (4.9)$$

This class of universal eigenfunctions is known from the early days of Calogero systems [8, 13]. It is easy to verify propositions 2.2 and 2.3 for this eigenfunction, since the classical limit is

$$\lim_{\hbar \rightarrow 0} \hbar^n L_n^{(\alpha)}(x) = (-1)^n \frac{\omega^n}{n!} (q^2 - \bar{q}^2)^n \quad (4.10)$$

in which $\omega \bar{q}^2 = \sum_{\rho \in \Delta_+} g_\rho$ [4].

The classical counterpart of the above general result (4.5), (4.6) is simply obtained as the $\hbar \rightarrow 0$ limit:

$$\hat{A}\varphi_N(q) = \omega N \varphi_N(q) \quad (4.11)$$

$$\varphi_N(q) \equiv \exp[\hat{A}_R/2\omega] \Phi_N(q) \quad \mathcal{D}\Phi_N(q) = N \Phi_N(q). \quad (4.12)$$

Since the operator \hat{A}_R (\hat{A}) satisfies the Leibnitz rule $\hat{A}_R(fg) = (\hat{A}_R f)g + f(\hat{A}_R g)$, we obtain corresponding to proposition 2.1 (2.19)

$$\exp[\hat{A}_R/2\omega] \Phi_N(q) \Phi_M(q) = \{\exp[\hat{A}_R/2\omega] \Phi_N(q)\} \{\exp[\hat{A}_R/2\omega] \Phi_M(q)\}. \quad (4.13)$$

The classical eigenfunctions of the elementary excitations are the generators of all the classical eigenfunctions. The quantum eigenfunctions of the elementary excitations play a less prominent role. The product of two quantum eigenfunctions is no longer a quantum eigenfunction, since the Laplacian Δ and thus the Hamiltonian \hat{H} do not enjoy the Leibnitz rule. Here we will show explicitly the classical and quantum eigenfunctions of the elementary excitations for the Calogero systems. Knowledge of the classical equilibrium and the eigenvectors of the Hessian matrix \tilde{W} (2.12) helps to determine the classical eigenfunctions.

4.1. A-series

Calogero and collaborators discussed the classical equilibrium problem of the A_r Calogero system about a quarter of a century ago [3, 14–16]. A modern version in terms of the prepotential was developed by Corrigan–Sasaki [4]. Following the usual convention we embed the root vectors in \mathbf{R}^{r+1} as

$$A_r = \{\mathbf{e}_j - \mathbf{e}_k, j, k = 1, \dots, r+1 | \mathbf{e}_j \in \mathbf{R}^{r+1}, \mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}\}. \quad (4.14)$$

The prepotentials for the full and *reduced* theories read

$$W = g \sum_{j < k}^{r+1} \log(q_j - q_k) - \frac{\omega}{2} q^2 \quad W = \sum_{j < k}^{r+1} \log(q_j - q_k) - \frac{1}{2} q^2. \quad (4.15)$$

We discuss the reduced theory for simplicity and brevity. Equations (2.10) determining the maximum of the ground state wavefunction ψ_0 read

$$\sum_{k \neq j}^{r+1} \frac{1}{\bar{q}_j - \bar{q}_k} = \bar{q}_j \quad j = 1, \dots, r+1. \quad (4.16)$$

These determine $\{\bar{q}_j\}$, $j = 1, \dots, r+1$ to be the zeros of the Hermite polynomial $H_{r+1}(x)$ [4, 14, 18],

$$H_{r+1}(\bar{q}_j) = 0.$$

The Hessian $-\tilde{W}$ has eigenvalues $\{1, 2, \dots, r+1\}$, which are exactly the quantum eigenvalues (divided by $\omega\hbar$) of the elementary excitations listed in table 1. (The lowest eigenvalue 1 belongs to the centre of mass degree of freedom which is completely decoupled from the other modes.)

Here are our new results on the classical and quantum eigenfunctions. The k th eigenvector of \tilde{W} has a simple form [5]

$$v_k = (P_k(\bar{q}_1), \dots, P_k(\bar{q}_{r+1})) \quad k = 0, \dots, r \quad (4.17)$$

in which $P_k(x)$ is a polynomial of degree k of a single variable. They obey the following three term recursion relation:

$$P_k(x) = xP_{k-1}(x) + \frac{k-r-2}{2}P_{k-2}(x) \quad P_0(x) = 1 \quad P_1(x) = x. \quad (4.18)$$

The orthogonality relations of the eigenvectors $\{v_k\}$ read simply

$$v_j \cdot v_k = 0 \iff \sum_{l=1}^{r+1} P_j(\bar{q}_l)P_k(\bar{q}_l) = 0 \quad j \neq k = 0, 1, \dots, r. \quad (4.19)$$

These are ‘orthogonal polynomials of a discrete variable’ [18, 19]. In the present case, the discrete variable is obviously the zeros of the Hermite polynomial.

A simple representation of the elementary excitations is provided by the elementary symmetric polynomials in $\{q_j\}$, $S_k(\{q_j\})$, $k = 0, 1, \dots, r+1$ (3.14) which are obviously Weyl invariant. By applying the operator \hat{A}_R (4.2) on the generating function (3.14)

$$G(x; \{q_j\}) = \prod_{j=1}^{r+1} (x + q_j) \quad (4.20)$$

and noting $\partial_{q_i} G = G/(x + q_i)$ and $\partial_x^2 G = \sum_{i,j} G/(x + q_i)(x + q_j)$, we obtain

$$\hat{A}_R G = \frac{1}{2} \partial_x^2 G \quad (4.21)$$

through partial fraction decomposition. This leads to

$$\hat{A} S_k(\{q_j\}) = k S_k + \frac{(r+3-k)(r+2-k)}{2} S_{k-2}(\{q_j\}). \quad (4.22)$$

We obtain the classical eigenfunctions for the elementary excitations:

$$\begin{aligned} \hat{A} \varphi_k(q) &= k \varphi_k(q) \\ \varphi_k(q) &= \exp[\hat{A}_R/2] S_k(\{q_j\}) = \sum_{l=0}^{[k/2]} \frac{(r+1-k+2l)!}{(r+1-k)!4^l l!} S_{k-2l}(\{q_j\}). \end{aligned} \quad (4.23)$$

Since φ_k are harmonic polynomials

$$\Delta S_k(\{q_j\}) = 0 \implies \Delta \varphi_k = 0 \quad k = 0, 1, \dots, r+1 \quad (4.24)$$

they are at the same time quantum eigenfunctions:

$$\hat{H}\varphi_k(q) = k\hbar\varphi_k(q) \quad k = 1, \dots, r + 1. \tag{4.25}$$

With some calculation one can verify propositions 2.2 and 2.3, that is $\varphi_k(\bar{q}) = 0$ and its derivative gives the above function P_k , $\partial/\partial q_j \varphi_k(\bar{q}) = (-)^k P_{k-1}(\bar{q}_j)$. Indeed, as $\partial_{q_j} S_k(\{q\}) = S_{k-1}(\{q\}) - q_j S_{k-2}(\{q\}) + \dots + (-q_j)^{k-1}$, one finds that

$$\begin{aligned} \partial_{q_j} \varphi_k(\bar{q}) &= (-)^{k-1} \left[\bar{q}_j^{k-1} + (S_2(\{\bar{q}\}) + (r+3-k)(r+2-k)/4) \bar{q}_j^{k-3} + \dots \right] \\ &\equiv (-)^{k-1} P_{k-1}(\bar{q}_j). \end{aligned} \tag{4.26}$$

The polynomial $P_k(x)$ of degree k (with $0 \leq k \leq r$) cannot vanish in all the $r + 1$ points \bar{q}_j (with $1 \leq j \leq r + 1$). Hence $\nabla \varphi_k(\bar{q}) \neq 0$ and it follows that $\varphi_k(q)$ is indeed an elementary excitation. As we now know that expressions (4.17) are eigenvectors corresponding to different eigenvalues of the matrix $-\tilde{W}$, it follows that the $P_k(x)$ are orthogonal polynomials of a discrete variable. Hence they obey a three term recurrence relation of type $P_k(x) = (A_k + B_k x)P_{k-1}(x) + C_k P_{k-2}(x)$. The coefficients of this recurrence are obtained from the definition (4.26) for different k , i.e. $A_k = 0$, $B_k = 1$, $C_k = (k-r-2)/2$.

4.2. B-series

Let us note that the rational C_r and BC_r systems are identical with the B_r system. The root vectors of B_r are expressed neatly in terms of an orthonormal basis of \mathbf{R}^r as

$$B_r = \{\pm \mathbf{e}_j \pm \mathbf{e}_k, \pm \mathbf{e}_j, j, k = 1, \dots, r | \mathbf{e}_j \in \mathbf{R}^r, \mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}\}. \tag{4.27}$$

The prepotentials for the full and *reduced* theories read

$$\begin{aligned} W &= g_L \sum_{j < k}^r \log(q_j^2 - q_k^2) + g_S \sum_{j=1}^r \log q_j - \frac{\omega}{2} q^2 \\ W &= \sum_{j < k}^r \log(q_j^2 - q_k^2) + \gamma \sum_{j=1}^r \log q_j - \frac{1}{2} q^2. \end{aligned} \tag{4.28}$$

We discuss the reduced theory. Assuming $\bar{q}_j \neq 0$, equations (2.10) determining the maximum of the ground state wavefunction ψ_0 read

$$\sum_{k \neq j}^r \frac{1}{\bar{q}_j^2 - \bar{q}_k^2} + \frac{\gamma/2}{\bar{q}_j^2} = \frac{1}{2} \quad j = 1, \dots, r \tag{4.29}$$

and determine $\{\bar{q}_j^2\}$, $j = 1, \dots, r$, as the zeros of the associated Laguerre polynomial $L_r^{(\gamma-1)}(x)$, [18, 3, 4], $L_r^{\gamma-1}(\bar{q}_j^2) = 0$. The Hessian $-\tilde{W}$ has eigenvalues $\{2, 4, 6, \dots, 2r\}$, which are exactly the quantum eigenvalues (divided by $\omega\hbar$) of the elementary excitations listed in table 1.

The new results on the classical and quantum eigenfunctions are as follows. The k th eigenvector of \tilde{W} has a simple form

$$v_{k-1} = (\bar{q}_1 P_{k-1}(\bar{q}_1^2), \dots, \bar{q}_r P_{k-1}(\bar{q}_r^2)) \quad k = 1, \dots, r \quad \bar{q}_l > 0 \tag{4.30}$$

in which the polynomials $\{P_k(x)\}$ obey the following three term recursion relation:

$$P_k(x) = (x - 2(r-k) - \gamma)P_{k-1}(x) - (k-r-\gamma)(k-r-1)P_{k-2}(x) \tag{4.31}$$

$$P_0(x) = 1 \quad P_1(x) = x - 2(r-1) - \gamma. \tag{4.32}$$

The orthogonality relations of the eigenvectors $\{v_k\}$ again correspond to those of orthogonal polynomials of a discrete variable:

$$v_j \cdot v_k = 0 \iff \sum_{l=1}^r \bar{q}_l^2 P_j(\bar{q}_l^2) P_k(\bar{q}_l^2) = 0 \quad j \neq k = 0, \dots, r-1. \quad (4.33)$$

A simple representation of the elementary excitations is provided by the elementary symmetric polynomials in $\{q_j^2\}$, $S_k(\{q_j^2\})$, $k = 0, 1, \dots, r$, which are obviously Weyl invariant. By applying the operator \hat{A}_R (4.2) and the Laplacian Δ on the generating function (3.14) $G(x; \{q_j^2\}) = \prod_{j=1}^r (x + q_j^2)$, we obtain

$$\hat{A}_R G = -2x \partial_x^2 G - 2\gamma \partial_x G \quad \text{and} \quad \Delta G = 2\partial_x G. \quad (4.34)$$

For the former formula, as in the A -series case, the partial fraction decomposition is used. These mean

$$\hat{A} S_k(\{q_j^2\}) = 2k S_k(\{q_j^2\}) + 2(r-k+1)(k-r-\gamma) S_{k-1}(\{q_j^2\}) \quad (4.35)$$

$$\Delta S_k(\{q_j^2\}) = 2(r-k+1) S_{k-1}(\{q_j^2\}) \quad k = 1, \dots, r \quad (4.36)$$

from which we obtain the classical eigenfunctions for the elementary excitations:

$$\hat{A} \phi_k(q) = 2k \phi_k(q) \quad k = 1, \dots, r \quad (4.37)$$

$$\phi_k(q) = \exp[\hat{A}_R/2] S_k(\{q_j^2\}) = \sum_{l=0}^k (-)^l \binom{r+l-k}{l} \frac{\Gamma(r+l-k+\gamma)}{\Gamma(r-k+\gamma)} S_{k-l}(\{q_j^2\}).$$

The corresponding quantum eigenfunctions have very similar forms, since the action of $-\hbar/2\Delta + \hat{A}$ on S_k (4.35), (4.36) is the same as that of \hat{A} with γ replaced by $\gamma + \hbar/2$:

$$\hat{H} \phi_k(q) = 2k\hbar \phi_k(q) \quad k = 1, \dots, r \quad (4.38)$$

$$\phi_k(q) = \exp[\hat{H}_R/2\hbar] S_k(\{q_j^2\}) = \sum_{l=0}^k (-)^l \binom{r+l-k}{l} \frac{\Gamma(r+l-k+\gamma+\hbar/2)}{\Gamma(r-k+\gamma+\hbar/2)} S_{k-l}(\{q_j^2\}).$$

The correspondence between the (classical) eigenfunctions (4.37), (4.38) and the classical eigenvectors (4.30) (and (4.31), (4.32)) is established in an identical manner as in the A -series case.

4.3. D -series

The root vectors of D_r are

$$D_r = \{\pm \mathbf{e}_j \pm \mathbf{e}_k, j, k = 1, \dots, r \mid \mathbf{e}_j \in \mathbf{R}^r, \mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}\}. \quad (4.39)$$

The prepotentials for the full and *reduced* theories read

$$W = g \sum_{j < k}^r \log(q_j^2 - q_k^2) - \frac{\omega}{2} q^2 \quad W = \sum_{j < k}^r \log(q_j^2 - q_k^2) - \frac{1}{2} q^2. \quad (4.40)$$

We discuss the reduced theory. The equilibrium position \bar{q}_j^2 are the zeros of the Laguerre polynomial $L_r^{-1}(x)$, that is one of $\{q_j^2\}$ is vanishing [4]. Let us choose $\bar{q}_1 = 0$.

The Hessian $-\tilde{W}$ has eigenvalues $\{2, 4, 6, \dots, 2(r-1), r\}$, which are exactly the quantum eigenvalues (divided by $\omega\hbar$) of the elementary excitations listed in table 1. The k th eigenvector

of \tilde{W} has a simple form

$$v_k = (0, \bar{q}_2 P_{k-1}(\bar{q}_2^2), \dots, \bar{q}_r P_{k-1}(\bar{q}_r^2)) \quad k = 1, \dots, r-1 \quad \bar{q}_l > 0 \quad (4.41)$$

$$v_r = (1, 0, \dots, 0) \quad (4.42)$$

in which the polynomials $\{P_k(x)\}$ obey the following three term recursion relation,

$$P_k(x) = (x - 2(r - k))P_{k-1}(x) - (k - r)(k - r - 1)P_{k-2}(x) \quad (4.43)$$

$$P_0(x) = 1 \quad P_1(x) = x - 2(r - 1) \quad (4.44)$$

which is a special case of that of B_r with $\gamma = 0$. The classical and quantum eigenfunctions $\varphi_1, \dots, \varphi_{r-1}$ and $\phi_1, \dots, \phi_{r-1}$ have the same form as in the B_r case (4.37), (4.38) with $\gamma = 0$. The special elementary excitation of the D_r theory belongs to the r th eigenvalue. The classical eigenfunction is the same as the quantum one

$$\varphi_r(q) = q_1 q_2 \cdots q_r \quad \hat{A}\varphi_r = r\varphi_r \quad \Delta\varphi_r = 0 \Rightarrow \hat{H}\varphi_r = r\hbar\varphi_r \quad (4.45)$$

$$\varphi_r(\bar{q}) = 0 \quad \nabla\varphi_r(q) |_{\bar{q}} \propto (1, 0, \dots, 0). \quad (4.46)$$

4.4. E-series

For the E-series of root systems we consider only the reduced theory, that is $\omega = g = 1$. The roots are normalized $\rho^2 = 2$, for all root systems E_6, E_7 and E_8 .

4.4.1. E_6 . The generating function is defined in terms of the weights belonging to the 27-dimensional representation:

$$G(x; \mathbf{27}) = \prod_{\mu \in \mathbf{27}} (x + \mu \cdot q) = \sum_{k=0}^{27} S_k x^{27-k}. \quad (4.47)$$

These are *minimal weights* having the properties $\rho \cdot \mu = \pm 1, 0$ for $\forall \rho \in E_6$ and

$$\mu^2 = 4/3 \quad \mu \cdot \nu = \begin{cases} 1/3 \\ -2/3 \end{cases} \quad \mu \neq \nu \in \mathbf{27}. \quad (4.48)$$

By applying the operator \hat{A}_R (4.2) and using the above properties of the weight vectors, we obtain

$$\hat{A}_R G = \frac{1}{2} \Delta G + \frac{1}{3} \partial_x^2 G \quad (4.49)$$

or in terms of the S_k ,

$$\hat{A}_R S_k = \frac{1}{2} \Delta S_k + \frac{1}{3} (29 - k)(28 - k) S_{k-2}. \quad (4.50)$$

Some lower members of S_k , which depend on $\{q_1, \dots, q_6\}$ are

$$S_0 = 1 \quad S_1 = 0 \quad S_2 = -3q^2 \quad S_3 = 0 \quad S_4 = \frac{15}{4}(q^2)^2 \quad \dots \quad (4.51)$$

The elementary excitations are for $N = 2, 5, 6, 8, 9, 12$ (see table 1) for which $\{S_N\}$ are functionally independent. The corresponding quantum eigenfunctions are

$$\phi_2 = q^2 - 3(12 + \hbar) \quad (4.52)$$

$$\phi_5 = S_5 \quad (4.53)$$

$$\phi_6 = S_6 + (1110 + 155\hbar)/4[(q^2)^2 - 4(9 + \hbar)(q^2 - (12 + \hbar))] \quad (4.54)$$

$$\begin{aligned} \phi_8 = S_8 + (57 + 13\hbar)S_6 - 15/2(1 - \hbar)(q^2)^3 \\ + 5(6 + \hbar)(2217 + 313\hbar)/24[3(q^2)^3 - 8(9 + \hbar)q^2 + 6(9 + \hbar)(12 + \hbar)] \end{aligned} \quad (4.55)$$

$$\phi_9 = S_9 - 35/6(15 + 4\hbar)S_5[q^2 - 2(9 + 2\hbar)] \quad (4.56)$$

$$\begin{aligned} \phi_{12} = S_{12} + \frac{(1725\,534 + 1146\,267\hbar + 188\,608\hbar^2)}{6144(3 + \hbar)(36 + 11\hbar)}(q^2)^6 + \frac{23}{576}S_5^2q^2 + \frac{101}{192}S_6^2 \\ + \frac{(222\,939 + 146\,112\hbar + 23\,798\hbar^2)}{960(3 + \hbar)(36 + 11\hbar)}(q^2)^3S_6 - \frac{(3216 + 937\hbar)}{1920(3 + \hbar)}(q^2)^2S_8. \end{aligned} \quad (4.57)$$

It should be noted that S_5 is a classical and quantum eigenfunction. The twelfth eigenfunction ϕ_{12} is chosen to be a homogeneous one.

4.4.2. E_7 . The generating function is defined in terms of the weights belonging to the **56**-dimensional representation:

$$G(x; \mathbf{56}) = \prod_{\mu \in \mathbf{56}} (x + \mu \cdot q) = \sum_{k=0}^{28} S_{2k} x^{56-2k}. \quad (4.58)$$

These are *minimal weights* having the properties $\rho \cdot \mu = \pm 1, 0$ for $\forall \rho \in E_7$ and

$$\mu \cdot \nu = \begin{cases} \pm 3/2 & \mu = \pm \nu \\ \pm 1/2 & \text{otherwise} \end{cases} \quad \mu, \nu \in \mathbf{56}. \quad (4.59)$$

Moreover **56** is *even*, i.e. if $\mu \in \mathbf{56}$ then $-\mu \in \mathbf{56}$. This is why the odd order terms in x vanish in (4.58). By applying the operator \hat{A}_R (4.2) and using the above properties of the weight vectors, we obtain

$$\hat{A}_R G = \frac{1}{2} \Delta G + \frac{1}{4} \partial_x^2 G + \frac{1}{2x} \partial_x G \quad (4.60)$$

or in terms of the S_{2k} ,

$$\hat{A}_R S_{2k} = \frac{1}{2} \Delta S_{2k} + \frac{1}{2}(29 - k)(59 - 2k)S_{2k-2}. \quad (4.61)$$

Some lower members of $S_{2k}(q_1, \dots, q_7)$ are

$$S_0 = 1 \quad S_2 = -6q^2 \quad S_4 = \frac{33}{2}(q^2)^2, \dots \quad (4.62)$$

The elementary excitations are for $N = 2, 6, 8, 10, 12, 14, 18$ (see table 1) for which $\{S_N\}$ are again functionally independent. The lower members of the corresponding quantum eigenfunctions are

$$\phi_2 = q^2 - \frac{7}{2}(18 + \hbar) \quad (4.63)$$

$$\phi_6 = S_6 + 3(6970 + 609\hbar)/4[(q^2)^2 - 9(14 + \hbar)/2(q^2 - 7(18 + \hbar)/6)] \quad (4.64)$$

$$\begin{aligned} \phi_8 = S_8 + (555/2 + 165\hbar/4)S_6 - 315(1 - \hbar)(q^2)^3 \\ + (536\,948 + 102\,416\hbar + 4851\hbar^2)45/32[(q^2)^2 - 3(14 + \hbar)(q^2 - 7(18 + \hbar)/8)]. \end{aligned} \quad (4.65)$$

4.4.3. E_8 . We have not succeeded in deriving an equation for a generating function similar to (4.21), (4.34), (4.49), (4.60). We start from a Weyl invariant power sum basis in terms of roots

$$\Phi_k(q_1, \dots, q_8) = \sum_{\rho \in \Delta_+} (\rho \cdot q)^k \quad k = 2, 8, 12 \dots \tag{4.66}$$

for those eight k listed in the E_8 row of table 1. They are functionally independent. For example, $\Phi_2 = 30q^2$. Although the derivation of the classical and quantum eigenfunctions for elementary excitations is straightforward, some results are too lengthy to present. For want of proper and convenient notation, we show only some lower members of the classical eigenfunctions:

$$\varphi_2 = q^2 - 120 \tag{4.67}$$

$$\varphi_8 = \Phi_8 - 197/750 \Phi_2^3 + 7092/5 \Phi_2^2 - 3404\,160 \Phi_2 + 3063\,744\,000 \tag{4.68}$$

$$\begin{aligned} \varphi_{12} = & \Phi_{12} - 1473/20 \Phi_8 \Phi_2 + 191/240\,000 \Phi_2^5 + 132\,570 \Phi_8 + 12\,551/5000 \Phi_2^4 \\ & - 118\,281/5 \Phi_2^3 + 63\,871\,740 \Phi_2^2 - 91\,975\,305\,600 \Phi_2 + 551\,851\,833\,360\,000. \end{aligned} \tag{4.69}$$

Verification of propositions 2.1–2.3 in section 2 is easy but the calculation is tedious.

4.5. F_4

The long roots of F_4 are the roots of D_4

$$\Delta_L = \{\pm \mathbf{e}_j \pm \mathbf{e}_k, j, k = 1, 2, 3, 4 \mid \mathbf{e}_j \in \mathbf{R}^4, \mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}\} \tag{4.70}$$

and the short roots are the union of vector, spinor and anti-spinor weights of D_4 :

$$\Delta_S = \{\pm \mathbf{e}_j \mid j = 1, 2, 3, 4\} \cup \{(\pm \mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4)/2\}. \tag{4.71}$$

We will consider a reduced theory in which $\omega = g_L = 1$ and the short root coupling is denoted by $g_S = \gamma$:

$$W = \sum_{\rho \in \Delta_{L+}} \log \rho \cdot q + \gamma \sum_{\rho \in \Delta_{S+}} \log \rho \cdot q - \frac{1}{2} q^2. \tag{4.72}$$

Let us introduce the elementary symmetric polynomials in $\{q_1^2, q_2^2, q_3^2, q_4^2\}$, as in the B -(D -) series (3.14):

$$\begin{aligned} S_1 &= q_1^2 + q_2^2 + q_3^2 + q_4^2 \equiv q^2 & S_2 &= q_1^2 q_2^2 + \dots + q_3^2 q_4^2 \\ S_3 &= q_1^2 q_2^2 q_3^2 + \dots + q_2^2 q_3^2 q_4^2 & S_4 &= q_1^2 q_2^2 q_3^2 q_4^2 \end{aligned} \tag{4.73}$$

which are not Weyl invariant, except for S_1 . A Weyl invariant basis for the elementary excitations is for degree 2, 6, 8 and 12 (see table 1) polynomials [17]:

$$\begin{aligned} \Phi_2 &= S_1 & \Phi_6 &= S_3 - S_1 S_2/6 & \Phi_8 &= S_4 - S_1 S_3/4 + S_2^2/12 \\ \Phi_{12} &= S_4 S_2 - S_2^3/36 - 3S_3^2/8 + S_1 S_2 S_3/8 - 3S_1^2 S_4/8. \end{aligned} \tag{4.74}$$

The quantum eigenfunctions for the elementary excitations are

$$\phi_2 = q^2 - 2(6(1 + \gamma) + \hbar) \tag{4.75}$$

$$\begin{aligned} \phi_6 &= \Phi_6 + (2(1 + \gamma) + \hbar)/4 \Phi_2^2 \\ & - (2(2 + \gamma) + \hbar)(4(1 + \gamma) + \hbar)/4 [3\Phi_2 - 2(6(1 + \gamma) + \hbar)] \end{aligned} \tag{4.76}$$

$$\begin{aligned} \phi_8 = & \Phi_8 + (3 + \hbar)\Phi_6 + (3 + \hbar)(2(2 + \gamma) + \hbar)/8\Phi_2^2 \\ & - (3 + \hbar)(2(2 + \gamma) + \hbar)(4(1 + \gamma) + \hbar)/8 [2\Phi_2 - (6(1 + \gamma) + \hbar)] \end{aligned} \quad (4.77)$$

$$\begin{aligned} \phi_{12} = & \Phi_{12} + (3 + 2\hbar)(2\Phi_8 + (3 + \hbar)\Phi_6)/2\Phi_2 - (3 + 2\hbar)(6(1 + \gamma) + 5\hbar)\Phi_8 \\ & + (3 + \hbar)(3 + 2\hbar)(2(2 + \gamma) + \hbar)/24\Phi_2^3 - (3 + \hbar)(3 + 2\hbar)(4(1 + \gamma) + 3\hbar)\Phi_6 \\ & - (3 + \hbar)(3 + 2\hbar)(2(1 + \gamma) + \hbar)(2(2 + \gamma) + \hbar)/16 \\ & \times [5\Phi_2^2 - 6(4(1 + \gamma) + \hbar)\Phi_2 + 2(4(1 + \gamma) + \hbar)(6(1 + \gamma) + \hbar)]. \end{aligned} \quad (4.78)$$

4.6. G_2 and dihedral root systems

The dihedral group of order $2m$, $I_2(m)$, is the group of orthogonal transformations that preserve a regular m -sided polygon in two dimensions. If all the roots are chosen to have the same length $\alpha_j^2 = 1$, they can be parametrized as

$$\alpha_j = (\cos(j\pi/m), \sin(j\pi/m)) \quad j = 1, \dots, 2m. \quad (4.79)$$

For odd m all of the roots are in the same orbit of the reflection group but for even m there are two orbits, one consisting of the α_j with odd j and the other with even j . Thus the $I_2(m)$ Calogero system has one coupling constant g for odd m and two couplings g_o and g_e for even m on top of the frequency ω of the harmonic confining potential. The complete set of quantum eigenfunctions ($\hbar = 1$) are given for all rank 2 Calogero systems in [8], with $A_2 \cong I_2(3)$, $B_2 \cong I_2(4)$ and $G_2 \cong I_2(6)$. So we concentrate on the elementary excitations with explicit \hbar dependence. The Coxeter invariant polynomials exist at degree 2 and m [8],

$$\Phi_2(q_1, q_2) = q^2 \quad \Phi_m(q_1, q_2) = \prod_{j=1}^m (v_j \cdot q) \quad (4.80)$$

where v_j are

$$v_j = (\cos((2j - 1)\pi/2m), \sin((2j - 1)\pi/2m)) \quad j = 1, \dots, m. \quad (4.81)$$

If we introduce the two-dimensional polar coordinate system⁵ for q , $q = r(\sin \theta, \cos \theta)$, the two Coxeter invariant polynomial variables read

$$\Phi_2 = q^2 = r^2 \quad \Phi_m = 2 \left(\frac{r}{2}\right)^m \cos m\theta. \quad (4.82)$$

The essential part of the quantum eigenfunctions is a Laguerre polynomial in r^2 , (4.9) times a Jacobi polynomial in $z = \cos m\theta$, thus the separation of variables is achieved.

As above, let us consider the reduced theory, $\omega = g = 1$ for odd m and $\omega = g_o = 1$ and $g_e = \gamma$ for even m . We have

$$\phi_2 = \begin{cases} \Phi_2 - (m + \hbar) & \text{odd } m \\ \Phi_2 - (m(1 + \gamma)/2 + \hbar) & \text{even } m \end{cases} \quad \Delta(\Phi_2)^l = 4l^2(\Phi_2)^{l-1} \quad (4.83)$$

and

$$\hat{A}_R \Phi_m = \begin{cases} 0 & \text{odd } m \\ (\gamma - 1)(\Phi_2)^{m/2-1} m^2 2^{-m} & \text{even } m \end{cases} \quad \Delta \Phi_m = 0. \quad (4.84)$$

Thus Φ_m is a classical and quantum eigenfunction for odd m . For even m we have

$$\phi_m = \Phi_m + \frac{(\gamma - 1)}{2^{m-1}(\gamma + 1 + \hbar)} r^m \quad (4.85)$$

⁵ We believe no confusion arises here between the radial coordinate variable r and the rank of the root system r , which in this case is 2 of $I_2(m)$.

which can be expressed as the Jacobi polynomial of degree 1 [8]. The classical equilibrium point is

$$(\bar{r}^2, \bar{\theta}) = \left(m, \frac{\pi}{2m}\right) \quad \left(m(1+\gamma)/2, \frac{2}{m} \arctan \sqrt{\gamma}\right) \quad (4.86)$$

for odd and even m , respectively. Verification of proposition 2.2 is straightforward.

4.7. H -series

The non-crystallographic Coxeter groups of H_3 and H_4 are the symmetry groups of the icosahedron and four-dimensional 600-cell, respectively. The former consists of 30 roots and the latter 120. Let us start from Coxeter invariant power sum bases in terms of roots

$$\Phi_k = \sum_{\rho \in \Delta_+} (\rho \cdot q)^k \quad k = 2, 6, 10 \text{ for } H_3 \quad k = 2, 12, 20, 30 \text{ for } H_4. \quad (4.87)$$

The quantum eigenfunctions of the elementary excitations in H_3 are

$$\phi_2 = q^2 - 3(10 + \hbar)/2 \quad (4.88)$$

$$\phi_6 = \Phi_6 - 15(13 + 3\hbar)/8[4(q^2)^2 - (6 + \hbar)(6q^2 - 3(10 + \hbar))] \quad (4.89)$$

$$\begin{aligned} \phi_{10} = \Phi_{10} - \frac{(215 + 126\hbar)}{12}[4q^2 - 15(2 + \hbar)]\Phi_6 + \frac{5}{6}(58 + 63\hbar)(q^2)^4 + \frac{75}{4}(109 + 45\hbar)(q^2)^3 \\ - \frac{25}{32}(10 + 3\hbar)(770 + 454\hbar + 63\hbar^2)[8(q^2)^2 - (6 + \hbar)(10q^2 - 3(10 + \hbar))]. \end{aligned} \quad (4.90)$$

Some of the quantum eigenfunctions of the elementary excitations in H_4 are

$$\phi_2 = q^2 - 2(30 + \hbar) \quad (4.91)$$

$$\begin{aligned} \phi_{12} = \Phi_{12} - \frac{315}{216}(565 + 66\hbar)[(q^2)^5 - 15(10 + \hbar)((q^2)^4 - 20/3(12 + \hbar) \\ \times ((q^2)^3 - 3(15 + \hbar)((q^2)^2 - 6/5(20 + \hbar)(q^2 - (30 + \hbar)/3))]. \end{aligned} \quad (4.92)$$

The other two eigenfunctions are too lengthy to be reported.

5. Classical and quantum eigenfunctions of the Sutherland systems

As shown in section 3 the eigenstates of the Sutherland system are specified by *dominant highest weight* $\lambda_{\vec{n}}$. The basis of the classical and quantum eigenfunctions is thus the sum of the exponentials of $2i\mu \cdot q$ taken for the entire Weyl orbit of $\lambda_{\vec{n}}$, $W \cdot \lambda_{\vec{n}}$, which we denote as

$$\Psi_{1^{n_1}2^{n_2}\dots r^{n_r}} = \sum_{\mu \in W \cdot \lambda_{\vec{n}}} e^{2i\mu \cdot q} \quad \lambda_{\vec{n}} = \sum_{j=1}^r n_j \lambda_j \quad n_j \in \mathbb{Z}_+. \quad (5.1)$$

As usual, if the multiplicity n_j is vanishing $n_j = 0$, it is not written. For example, the basis corresponding to the fundamental weights $\lambda_1, \dots, \lambda_r$ are $\Psi_1, \Psi_2, \dots, \Psi_r$:

$$\Psi_1 = \sum_{\mu \in W \cdot \lambda_1} e^{2i\mu \cdot q}, \dots, \Psi_r = \sum_{\mu \in W \cdot \lambda_r} e^{2i\mu \cdot q}. \quad (5.2)$$

The operator \hat{A} is lower triangular and the Laplacian [8, 20] is diagonal in this basis:

$$\hat{A}\Psi_\lambda = 4Q \cdot \lambda \Psi_\lambda + \sum_{|\lambda'| < |\lambda|} c_{\lambda'} \Psi_{\lambda'} \quad \Delta \Psi_\lambda = -4\lambda^2 \Psi_\lambda. \quad (5.3)$$

As shown below there is a marked difference in the forms of the eigenfunctions between the classical (A , B , C and D) and the exceptional (E , F_4 and G_2) root systems. The elementary excitations of the A -series Sutherland system can have the same classical and quantum eigenfunctions as in the Calogero case.

5.1. A -series

The classical equilibrium of the A_r Sutherland system is rather trivial [15, 16]. The prepotentials for the full and *reduced* theories read

$$W = g \sum_{j < k}^{r+1} \log \sin(q_j - q_k) \quad W = \sum_{j < k}^{r+1} \log \sin(q_j - q_k). \quad (5.4)$$

Equations (2.10) determining the maximum of the ground state wavefunction ψ_0 read

$$\sum_{k \neq j}^{r+1} \cot[\bar{q}_j - \bar{q}_k] = 0 \quad j = 1, \dots, r+1$$

which are satisfied by the *equally spaced* configuration $\bar{q}_j = (2j - (r+2))\pi/2(r+1)$. The Hessian $-\bar{W}$ has eigenvalues $2\{r, (r-1)2, \dots, (r+1-k)k, \dots, 2(r-1), r\}$, which can be expressed as $\{4\delta \cdot \lambda_1, \dots, 4\delta \cdot \lambda_k, \dots, 4\delta \cdot \lambda_r\}$ with δ defined in (3.12). In these formulae the trivial eigenvalue 0, coming from the translational invariance, is removed. The k th eigenvector of \bar{W} is simply $v_k = (e^{2ik\bar{q}_1}, \dots, e^{2ik\bar{q}_{r+1}})$. The orthogonality condition of the eigenvectors $\{v_k\}$ read simply $v_j \cdot v_k = \sum_l e^{2i(j+k)\bar{q}_l} = 0$.

Let us introduce a generating function

$$G(x; \{e^{2iq_j}\}) = \prod_{j=1}^{r+1} (x + e^{2iq_j}) = \sum_{k=0}^{r+1} S_k(\{e^{2iq_j}\}) x^{r+1-k}. \quad (5.5)$$

It is easy to see that the symmetric polynomial S_k is equal to the basis Ψ_k (5.2) up to a term proportional to the ‘centre of mass’ $q_1 + \dots + q_{r+1}$ which is orthogonal to all the A_r roots. The generating function G satisfies

$$\hat{A}G = 2rx\partial_x G - 2x^2\partial_x^2 G \quad \Delta G = -4(r+1)G + 4x\partial_x G \quad (5.6)$$

which translate into

$$\hat{A}S_k = 2k(r+1-k)S_k \quad \Delta S_k = -4kS_k. \quad (5.7)$$

Therefore

$$\varphi_k = S_k(\{e^{2iq_j}\}) \quad k = 1, \dots, r \quad (5.8)$$

is a classical and quantum eigenfunction of the k th elementary excitation with eigenvalues $2k(r+1-k)$ and $2k\hbar(r+1-k+\hbar)$ of \hat{A} (2.16) and \hat{H} (2.15), respectively. The absence of quantum corrections is a general property shared by eigenfunctions belonging to *minimal weights* [8]. All the fundamental representations of the A -series root systems are minimal.

5.2. B - and C -series

Since the B and C root systems are closely related, $B \leftrightarrow C$ for $\alpha \leftrightarrow \alpha^\vee = 2\alpha/\alpha^2$, many formulae for the eigenfunctions etc take similar forms. It is advantageous to write these expressions in parallel so that the similarity and differences can be well appreciated. The prepotentials for the *reduced* theory read

$$W = \sum_{j < k}^r \log(\cos 2q_j - \cos 2q_k) + \gamma \sum_{j=1}^r \log \sin q_j \quad B\text{-series} \quad (5.9)$$

$$W = \sum_{j < k}^r \log(\cos 2q_j - \cos 2q_k) + \gamma \sum_{j=1}^r \log \sin 2q_j \quad C\text{-series.} \quad (5.10)$$

Equations (2.10) determining the maximum of the ground state wavefunction ψ_0 read

$$\sum_{k \neq j}^r \frac{1}{\bar{x}_j - \bar{x}_k} + \frac{\gamma}{2} \frac{1}{\bar{x}_j - 1} = 0 \quad (B) \quad J = 1, \dots, r \quad (5.11)$$

$$\sum_{k \neq j}^r \frac{1}{\bar{x}_j - \bar{x}_k} + \frac{\gamma}{2} \frac{1}{\bar{x}_j - 1} + \frac{\gamma}{2} \frac{1}{\bar{x}_j + 1} = 0 \quad (C) \quad J = 1, \dots, r \quad (5.12)$$

for $\bar{x}_j = \cos 2\bar{q}_j$. They determine $\{\bar{x}_j\}$ as the zeros of Jacobi polynomials [4]:

$$P_r^{(\gamma-1, -1)}(\bar{x}_j) = 0 \quad (B) \quad P_r^{(\gamma-1, \gamma-1)}(\bar{x}_j) = 0 \quad (C).$$

Because of the identity

$$P_r^{(a, -1)}(x) = \frac{a+n}{2n} (x+1) P_{r-1}^{(a, 1)}(x)$$

the (B) case always has one zero at $\bar{x} = -1$. Let us choose $\bar{x}_r = -1 \Leftrightarrow \bar{q}_r = \pi/2$. The Hessian $-\tilde{W}$ has eigenvalues

$$2k(2r - k + \gamma - 1) \quad k = 1, \dots, r-1, \quad r(r + \gamma - 1) \quad (B) \quad (5.13)$$

$$2k(2r - k + 2\gamma - 1) \quad k = 1, \dots, r \quad (C) \quad (5.14)$$

in which the last one of the B-series belongs to the spinor representation (λ_r) . The k th eigenvector of \tilde{W} has the form

$$v_k = (\sin 2\bar{q}_1 P_{k-1}(\bar{x}_1), \dots, \sin 2\bar{q}_{r-1} P_{k-1}(\bar{x}_{r-1}), 0) \quad k = 1, \dots, r-1 \quad (r) \quad (B \text{ and } C) \quad (5.15)$$

$$v_r = (0, 0, \dots, 1) \quad (B) \quad (5.16)$$

in which the polynomials $\{P_k\}$ of a single variable x obey the three term recursion relation,

$$P_0(x) = 1 \quad P_1(x) = x + \frac{\gamma(\gamma - 2)}{(2r + \gamma - 2)(2r + \gamma - 4)} \quad (5.17)$$

$$P_k(x) = \left(x + \frac{\gamma(\gamma - 2)}{(2r + \gamma - 2k)(2r - 2k + \gamma - 2)} \right) P_{k-1}(x) - \frac{4(r-k)(r-k+1)(r-k+\gamma)(r-k+\gamma-1)}{(2r-2k+\gamma)^2(2r-2k+\gamma+1)(2r-2k+\gamma-1)} P_{k-2}(x) \quad (5.18)$$

for the B-series and

$$P_0(x) = 1 \quad P_1(x) = x \quad (5.19)$$

$$P_k(x) = x P_{k-1}(x) - \frac{(r-k+1)(r-k+2\gamma-1)}{(2r-2k+2\gamma+1)(2r-2k+2\gamma-1)} P_{k-2}(x) \quad (5.20)$$

for the C-series. The orthogonality conditions for these discrete variable polynomials are

$$\sum_{j=1}^r (1 - \bar{x}_j^2) P_k(\bar{x}_j) P_l(\bar{x}_j) = \delta_{kl} \quad (5.21)$$

with $\{\bar{x}_j\}$ being the zeros of a Jacobi polynomial.

Let us introduce a generating function

$$G(x; \{\cos 2q_j\}) = \prod_{j=1}^r (x + \cos 2q_j) = \sum_{k=0}^r S_k(\{\cos 2q_j\}) x^{r-k}. \quad (5.22)$$

It is easy to see that the symmetric polynomial S_k is proportional to the basis Ψ_k (5.2):

$$S_k(\{\cos 2q_j\}) = 2^{-k} \Psi_k \quad k = 1, \dots, r-1, \quad (B \text{ and } C) \quad (5.23)$$

$$\Psi_r = \sum_{\mu: \text{spinor weights}} e^{2i\mu \cdot q} = 2^r \prod_{j=1}^r \cos q_j \quad (B). \quad (5.24)$$

The generating function satisfies

$$\hat{A}G = 2r(r + \gamma - 1)G + 2(1 - x^2)\partial_x^2 G + 2\gamma(1 - x)\partial_x G \quad (B) \quad (5.25)$$

$$\hat{A}G = 2r(r + 2\gamma - 1)G + 2(1 - x^2)\partial_x^2 G - 4\gamma x \partial_x G \quad (C) \quad (5.26)$$

$$\Delta G = -4rG + 4x \partial_x G \quad (B \text{ and } C). \quad (5.27)$$

These mean in turn

$$\hat{A}S_k = 2k(2r - k + \gamma - 1)S_k + 2\gamma(r - k + 1)S_{k-1} + 2(r + 1 - k)(r + 2 - k)S_{k-2} \quad (B) \quad (5.28)$$

$$\hat{A}\Psi_r = r(r + \gamma - 1)\Psi_r \quad (B) \quad (5.29)$$

$$\hat{A}S_k = 2k(2r - k + 2\gamma - 1)S_k + 2(r + 1 - k)(r + 2 - k)S_{k-2} \quad (C) \quad (5.30)$$

$$\Delta S_k = -4kS_k \quad \Delta \Psi_r = -r\Psi_r \quad (B \text{ and } C). \quad (5.31)$$

The quantum eigenfunctions for the elementary excitations are

$$\phi_k = S_k + \frac{\gamma(r - k + 1)}{2r - 2k + \hbar + \gamma} S_{k-1} + \frac{(r - k - 1)(r - k + 2)(2r - 2k + \hbar + \gamma + \gamma^2)}{2(2r - 2k + \hbar + \gamma)(2r - 2k + \hbar + \gamma + 1)} S_{k-2} + \dots \quad k = 1, \dots, r-1 \quad (5.32)$$

$$\phi_r = \prod_{j=1}^r \cos q_j \quad (B) \quad (5.33)$$

$$\phi_k = \sum_{l=0}^{[k/2]} \frac{(r + 2l - k)! \Gamma(r - k + \gamma + \hbar/2 + 1/2)}{4^l l! (r - k)! \Gamma(r - k + \gamma + \hbar/2 + l + 1/2)} S_{k-2l} \quad k = 1, \dots, r \quad (C). \quad (5.34)$$

The corresponding eigenvalues are

$$2k\hbar(2r - k + \gamma - 1 + \hbar) \quad k = 1, \dots, r-1, \quad r\hbar(r + \gamma + \hbar/2 - 1) \quad (B) \quad (5.35)$$

$$2k\hbar(2r - k + 2\gamma - 1 + \hbar) \quad k = 1, \dots, r \quad (C). \quad (5.36)$$

The eigenfunction for the spinor weight in B -series ϕ_r (5.33) has no quantum corrections. The representations (5.15), (5.16) and the recursions (5.17)–(5.20) are obtained from these eigenfunctions.

5.3. *D-series*

The reduced prepotential of the *D*-series Sutherland system is obtained by removing the short (long) root coupling γ terms from those of the *B*- and *C*- series (5.9), (5.10). This results in the change of the eigenvectors $\{v_k\}$ (5.15), (5.16) \rightarrow (5.38), (5.39) and the emergence of another eigenfunction associated with anti-spinor weights ϕ_{r-1} (5.50) which receives no quantum corrections. Equations (2.10) determining the maximum of the ground state wavefunction ψ_0 have a solution $\bar{q}_1 = 0$, $\bar{q}_r = \pi/2$ and with $\bar{x}_j = \cos 2\bar{q}_j$, $j = 2, \dots, r-1$ being the zeros of the Jacobi polynomial $P_{r-2}^{(1,1)}(x)$ or equivalently of the Gegenbauer polynomial $C_{r-2}^{3/2}(x)$:

$$P_{r-2}^{(1,1)}(\bar{x}_j) = 0 \quad C_{r-2}^{3/2}(\bar{x}_j) = 0 \quad j = 2, \dots, r-1.$$

The Hessian $-\tilde{W}$ has eigenvalues

$$2k(2r-k-1) \quad k = 1, \dots, r-2, \quad r(r-1) [2] \quad (5.37)$$

in which the exceptional one is doubly degenerate corresponding to the ‘fish tail’ of the *D*-series Dynkin diagram. The corresponding eigenvectors of \tilde{W} are

$$v_k = (0, \sin 2\bar{q}_2 P_{k-1}(\bar{x}_2), \dots, \sin 2\bar{q}_{r-1} P_{k-1}(\bar{x}_{r-1}), 0) \quad k = 1, \dots, r-2 \quad (5.38)$$

$$v_{r-1} = (1, 0, \dots, 0) \quad v_r = (0, 0, \dots, 1). \quad (5.39)$$

The polynomials $\{P_k\}$ of a single variable x obey simple three term recursion relations:

$$P_0(x) = 1 \quad P_1(x) = x \quad (5.40)$$

$$P_k(x) = xP_{k-1}(x) - \frac{(r-k+1)(r-k-1)}{4(r-k+1/2)(r-k-1/2)} P_{k-2}(x). \quad (5.41)$$

The generating function has the same form as in the *B*, *C* cases:

$$G(x; \{\cos 2q_j\}) = \prod_{j=1}^r (x + \cos 2q_j) = \sum_{k=0}^r S_k(\{\cos 2q_j\}) x^{r-k}.$$

It is easy to see that the symmetric polynomial S_k is proportional to the basis Ψ_k (5.2) and that the two additional bases are

$$S_k(\{\cos 2q_j\}) = 2^{-k} \Psi_k \quad k = 1, \dots, r-2 \quad (5.42)$$

$$\Psi_{r-1} \propto \prod_{j=1}^r \sin q_j \quad \Psi_r \propto \prod_{j=1}^r \cos q_j. \quad (5.43)$$

They satisfy

$$\hat{A}G = 2r(r-1)G + 2(1-x^2)\partial_x^2 G \quad (5.44)$$

$$\hat{A}\Psi_{r-1} = r(r-1)\Psi_{r-1} \quad \hat{A}\Psi_r = r(r-1)\Psi_r \quad (5.45)$$

$$\Delta G = -4rG + 4x\partial_x G \quad \Delta\Psi_{r-1} = -r\Psi_{r-1} \quad \Delta\Psi_r = -r\Psi_r. \quad (5.46)$$

These imply for S_k

$$\hat{A}S_k = 2k(2r-k-1)S_k + 2(r+1-k)(r+2-k)S_{k-2} \quad (5.47)$$

$$\Delta S_k = -4kS_k. \quad (5.48)$$

Thus we arrive at the quantum eigenfunctions corresponding to the elementary excitations,

$$\phi_k = \sum_{l=0}^{[k/2]} \frac{(r+2l-k)!\Gamma(r-k+\hbar/2+1/2)}{4^l l! (r-k)!\Gamma(r-k+\hbar/2+l+1/2)} S_{k-2l} \quad k = 1, \dots, r-2 \quad (5.49)$$

$$\phi_{r-1} = \prod_{j=1}^r \sin q_j \quad \phi_r = \prod_{j=1}^r \cos q_j \quad (5.50)$$

with the eigenvalues of \hat{H}

$$2k\hbar(2r - k - 1 + \hbar) \quad k = 1, \dots, r - 2, \quad r\hbar(r + \hbar/2 - 1) \quad [2]. \quad (5.51)$$

Again, the eigenfunctions corresponding to the spinor and anti-spinor weights receive no quantum corrections. These are minimal weights.

5.4. E-series

For the Sutherland systems based on exceptional root systems, E , F and G , the method of the generating functions seems not so useful as in the classical root system cases, because of the ‘exceptional’ character. The equilibrium points of the potentials are not related to known classical polynomials in contrast to the cases discussed above. New polynomials describing the equilibria were introduced by Odake and Sasaki [4]. Here we will construct the eigenfunctions corresponding to the fundamental weights (elementary excitations) starting from the basis Ψ_1, \dots, Ψ_r (5.2). There is no universally accepted way of naming the simple roots and fundamental weights of the exceptional root systems. We show our convention in terms of the Dynkin diagrams.

5.4.1. E_6 . The symmetry of the Dynkin diagram (see figure 1) is reflected in the structure of the eigenvalues and eigenfunctions, too. The spectra of the Hessian $-\tilde{W}$ and the corresponding \hat{H} in the order of $\lambda_1, \dots, \lambda_6$ are

$$\{32, 44, 60, 84, 60, 32\} \quad (5.52)$$

$$\left\{ \frac{8}{3}\hbar(12 + \hbar), 4\hbar(11 + \hbar), \frac{20}{3}\hbar(9 + \hbar), 12\hbar(7 + \hbar), \frac{20}{3}\hbar(9 + \hbar), \frac{8}{3}\hbar(12 + \hbar) \right\}. \quad (5.53)$$

The quantum eigenfunctions are listed in the order of increasing energy eigenvalues and the values of the λ_j^2 :

$$4/3: \quad \phi_1 = \Psi_1 \quad \phi_6 = \Psi_6 \quad (5.54)$$

$$2: \quad \phi_2 = \Psi_2 + \frac{72}{(11 + \hbar)} \quad (5.55)$$

$$10/3: \quad \phi_3 = \Psi_3 + \frac{40}{(7 + \hbar)}\Psi_6 \quad \phi_5 = \Psi_5 + \frac{40}{(7 + \hbar)}\Psi_1 \quad (5.56)$$

$$6: \quad \phi_4 = \Psi_4 + \frac{24}{(5 + \hbar)}\Psi_{16} + \frac{30(17 + \hbar)}{(5 + \hbar)^2}\Psi_2 + \frac{720(17 + \hbar)}{(5 + \hbar)^2(7 + \hbar)}. \quad (5.57)$$

The orbits in sub-leading terms of an eigenfunction are contained in the *Lie algebra representation* specified by the dominant weight of the leading term. Constant terms correspond to zero weights. For example, the highest weight representation specified by λ_4 (5.57) consists of the Weyl orbits of $\lambda_4, \lambda_1 + \lambda_6, \lambda_2$ and some zero weights. The lowest two eigenfunctions (5.54) consist of single orbits belonging to the **27** and $\overline{\mathbf{27}}$ representations, which are minimal. Thus they do not receive quantum corrections. They correspond to the left and right ends of the diagram, figure 1. The fundamental weight λ_2 (5.55) corresponds to the adjoint representation, containing all the roots and the rank number of zero weights. The constant term in (5.55), $72/(11 + \hbar)$, reflects the number of roots 72 and the highest exponent 11 which is the ‘height’ of the *highest root*, that is λ_2 in the present case [8]. The longer the dominant weight $\lambda_{\frac{2}{n}}$ becomes, the more complicated structure the corresponding eigenfunction has. These are common features of all the eigenfunctions of the Sutherland systems.

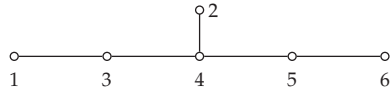


Figure 1. E_6 Dynkin diagram with the numbers of the simple roots attached.

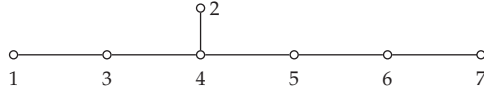


Figure 2. E_7 Dynkin diagram with the numbers of the simple roots attached.

5.4.2. E_7 . The spectra of the Hessian $-\tilde{W}$ and the corresponding \hat{H} are

$$\{68, 98, 132, 192, 150, 104, 54\} \tag{5.58}$$

$$\{4\hbar(17 + \hbar), 7\hbar(14 + \hbar), 12\hbar(11 + \hbar), 24\hbar(8 + \hbar), 15\hbar(10 + \hbar), 8\hbar(13 + \hbar), 3\hbar(18 + \hbar)\} \tag{5.59}$$

which have no degeneracy. The Dynkin diagram (see figure 2) has no symmetry. The corresponding quantum eigenfunctions and the values of the λ_j^2 are

$$3/2: \quad \phi_7 = \Psi_7 \tag{5.60}$$

$$2: \quad \phi_1 = \Psi_1 + \frac{126}{(17 + \hbar)} \tag{5.61}$$

$$7/2: \quad \phi_2 = \Psi_2 + \frac{72}{(11 + \hbar)} \Psi_7 \tag{5.62}$$

$$4: \quad \phi_6 = \Psi_6 + \frac{60}{(9 + \hbar)} \Psi_1 + \frac{3780}{(9 + \hbar)(13 + \hbar)} \tag{5.63}$$

$$6: \quad \phi_3 = \Psi_3 + \frac{40}{(7 + \hbar)} \Psi_6 + \frac{48(32 + \hbar)}{(7 + \hbar)(8 + \hbar)} \Psi_1 + \frac{2016(32 + \hbar)}{(7 + \hbar)(8 + \hbar)(11 + \hbar)} \tag{5.64}$$

$$15/2: \quad \phi_5 = \Psi_5 + \frac{40}{(7 + \hbar)} \Psi_{17} + \frac{105(23 + \hbar)}{(7 + \hbar)(13 + 2\hbar)} \Psi_2 + \frac{360(200 + 13\hbar)}{(7 + \hbar)(8 + \hbar)(13 + 2\hbar)} \Psi_7 \tag{5.65}$$

$$\begin{aligned} 12: \quad \phi_4 = & \Psi_4 + \frac{24}{(5 + \hbar)} \Psi_{16} + \frac{30(17 + \hbar)}{(5 + \hbar)^2} \Psi_{27} + \frac{720}{(5 + \hbar)(7 + \hbar)} \Psi_{12} \\ & + \frac{720(17 + \hbar)}{(5 + \hbar)^2(7 + \hbar)} \Psi_{72} + \frac{20(3340 + 911\hbar + 68\hbar^2 + \hbar^3)}{(5 + \hbar)^3(7 + \hbar)} \Psi_3 \\ & + \frac{40(59\,325 + 19\,900\hbar + 2126\hbar^2 + 80\hbar^3 + \hbar^4)}{(5 + \hbar)^3(7 + \hbar)(11 + 2\hbar)} \Psi_6 \\ & + \frac{480(14\,735 + 3289\hbar + 223\hbar^2 + 5\hbar^3)}{(5 + \hbar)^3(7 + \hbar)(11 + 2\hbar)} \Psi_1 + \frac{10\,080(1945 + 228\hbar + 11\hbar^2)}{(5 + \hbar)^3(7 + \hbar)(11 + 2\hbar)}. \end{aligned} \tag{5.66}$$

The first corresponds to the **56**-dimensional representation which is minimal. It has no \hbar dependence. The second corresponds to the set of roots (adjoint representation) with 126

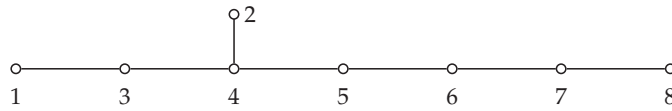


Figure 3. E_8 Dynkin diagram with the numbers of the simple roots attached.

roots and the highest exponent being 17. The last expression (5.66) is much longer than its classical counterpart

$$\varphi_4 = \Psi_4 + \frac{24}{5}\Psi_{16} + \frac{102}{5}\Psi_{27} + \frac{144}{7}\Psi_{12} + \frac{2448}{35}\Psi_{72} + \frac{2672}{35}\Psi_3 + \frac{2712}{11}\Psi_6 + \frac{40416}{55}\Psi_1 + \frac{112032}{55}. \tag{5.67}$$

For the E_8 eigenfunctions, we will present the classical ones simply because of the lack of space.

5.4.3. E_8 . The spectra of the Hessian $-\tilde{W}$ and the corresponding \hat{H} in the order of $\lambda_1, \dots, \lambda_8$ are

$$\{184, 272, 364, 540, 440, 336, 228, 116\} \tag{5.68}$$

$$\{8\hbar(23 + \hbar), 16\hbar(17 + \hbar), 28\hbar(13 + \hbar), 60\hbar(9 + \hbar), 40\hbar(11 + \hbar), 24\hbar(14 + \hbar), 12\hbar(19 + \hbar), 4\hbar(29 + \hbar)\} \tag{5.69}$$

which has no degeneracy. The Dynkin diagram (see figure 3) has no symmetry.

The classical eigenfunctions for six lower elementary excitations are

$$2: \varphi_8 = \Psi_8 + \frac{240}{29} \tag{5.70}$$

$$4: \varphi_1 = \Psi_1 + \frac{126}{17}\Psi_8 + \frac{15\,120}{17 \cdot 23} \tag{5.71}$$

$$6: \varphi_7 = \Psi_7 + \frac{84}{11}\Psi_1 + \frac{444}{11}\Psi_8 + \frac{35\,520}{11 \cdot 19} \tag{5.72}$$

$$8: \varphi_2 = \Psi_2 + \frac{72}{11}\Psi_7 + \frac{4080}{11^2}\Psi_1 + \frac{215\,712}{11^2 \cdot 13}\Psi_8 + \frac{12\,942\,720}{11^2 \cdot 13 \cdot 17} \tag{5.73}$$

$$12: \varphi_6 = \Psi_6 + \frac{20}{3}\Psi_{18} + \frac{420}{13}\Psi_{8^2} + \frac{203}{6}\Psi_2 + \frac{1776}{13}\Psi_7 + \frac{361\,004}{3 \cdot 13 \cdot 19}\Psi_1 + \frac{4255\,608}{11 \cdot 13 \cdot 19}\Psi_8 + \frac{12\,660\,480}{11 \cdot 13 \cdot 19} \tag{5.74}$$

$$14: \varphi_3 = \Psi_3 + \frac{40}{7}\Psi_6 + \frac{192}{7}\Psi_{18} + \frac{1152}{11}\Psi_{8^2} + \frac{2608}{23}\Psi_2 + \frac{12\,023\,496}{7 \cdot 11 \cdot 17 \cdot 23}\Psi_7 + \frac{5525\,664}{11 \cdot 17 \cdot 23}\Psi_1 + \frac{16\,392\,384}{11 \cdot 17 \cdot 23}\Psi_8 + \frac{592\,911\,360}{11 \cdot 13 \cdot 17 \cdot 23}. \tag{5.75}$$

Most of the denominators contain the exponents of E_8 , $\{1, 7, 11, 13, 17, 19, 23, 29\}$. This is a common feature shared by all the root systems but seen most clearly in the exceptional root system cases. The two unlisted eigenfunctions φ_5 and φ_4 have simply too many terms to be presented here. For φ_5 , $\lambda_5^2 = 20$, the number of elements in the Weyl orbit of λ_5 is 241 920 and the highest weight representation of λ_5 is 146 325 270 dimensional. The eigenfunction φ_5



Figure 4. F_4 Dynkin diagram with the numbers of the simple roots attached.

contains 14 terms corresponding to the dominant characters in the Lie algebra representation of the highest weight λ_5 , that is $\lambda_5, \lambda_1 + \lambda_7, 2\lambda_1, \lambda_2 + \lambda_8, \lambda_7 + \lambda_8, \lambda_3, \lambda_6, \lambda_1 + \lambda_8, 2\lambda_8, \lambda_2, \lambda_7, \lambda_1, \lambda_8$ and zero weights. For $\varphi_4, \lambda_4^2 = 30$, the number of elements in the Weyl orbit of λ_4 is 483 840 and the highest weight representation of λ_4 is 6899 079 264 dimensional. The eigenfunction φ_4 contains 24 terms and some of their coefficients are ratios of enormously large integers.

5.5. F_4 and G_2

The Sutherland systems based on F_4 and G_2 are interesting because of the interplay of the long and short root couplings. While we show the Dynkin diagram of F_4 to indicate our convention for simple root naming, we simply agree that α_1 is the short simple root of G_2 , thus α_2 is the long simple root.

The spectra of the Hessian $-\tilde{W}$ and the corresponding \hat{H} of F_4 (see figure 4) in the order of $\lambda_1, \dots, \lambda_4$ are

$$\{20 + 12\gamma, 36 + 24\gamma, 24 + 18\gamma, 12 + 10\gamma\} \tag{5.76}$$

$$\{4\hbar(5 + 3\gamma + \hbar), 12\hbar(3 + 2\gamma + \hbar), 6\hbar(4 + 3\gamma + \hbar), 2\hbar(6 + 5\gamma + \hbar)\}. \tag{5.77}$$

The quantum eigenfunctions are listed in the order of increasing energy eigenvalues and the values of the λ_j^2 :

$$1: \quad \phi_4 = \Psi_4 + \frac{24\gamma}{6 + 5\gamma + \hbar} \tag{5.78}$$

$$2: \quad \phi_1 = \Psi_1 + \frac{6\gamma}{4 + \gamma + \hbar} \Psi_4 + \frac{24(4 + \gamma + 3\gamma^2 + \hbar)}{(4 + \gamma + \hbar)(5 + 3\gamma + \hbar)} \tag{5.79}$$

$$3: \quad \phi_3 = \Psi_3 + \frac{12\gamma}{2 + 3\gamma + \hbar} \Psi_1 + \frac{12(2 + 5\gamma + 6\gamma^2 + \hbar + \hbar\gamma)}{(3 + 2\gamma + \hbar)(2 + \gamma + \hbar)} \Psi_4 + \frac{96\gamma(8 + 9\gamma + 6\gamma^2 + 3\hbar + \hbar\gamma)}{(3 + 2\gamma + \hbar)(2 + 3\gamma + \hbar)(4 + 3\gamma + \hbar)} \tag{5.80}$$

$$6: \quad \phi_2 = \Psi_2 + \frac{4\gamma}{2 + \gamma} \Psi_{14} + \frac{12(2 + \gamma + \gamma^2)}{(2 + \gamma)(3 + \gamma)} \Psi_{42} + \frac{4\gamma(11 + 9\gamma)}{(2 + \gamma)(3 + \gamma)} \Psi_3 + 12[24 + 22\gamma + 47\gamma^2 + 23\gamma^3][(2 + \gamma)(3 + \gamma)(4 + 3\gamma)]^{-1} \Psi_1 + 24\gamma[28 + 37\gamma + 27\gamma^2][(2 + \gamma)(3 + \gamma)(4 + 3\gamma)]^{-1} \Psi_4 + 96[24 + 30\gamma + 85\gamma^2 + 67\gamma^3 + 30\gamma^4][(2 + \gamma)(3 + \gamma)(4 + 3\gamma)(3 + 2\gamma)]^{-1}. \tag{5.81}$$

Here we listed the classical eigenfunction φ_2 (5.81) for the highest elementary excitation, simply for display reasons.

The spectra of the Hessian $-\tilde{W}$ and the corresponding \hat{H} of G_2 in the order of λ_1, λ_2 are

$$\left\{4 + \frac{8}{3}\gamma, 8 + 4\gamma\right\} \quad \left\{\frac{4}{3}\hbar(3 + 2\gamma + \hbar), 4\hbar(2 + \gamma + \hbar)\right\}. \tag{5.82}$$

The quantum eigenfunctions are listed in the order of increasing energy eigenvalues and the values of the λ_j^2 :

$$2/3: \quad \phi_1 = \Psi_1 + \frac{6\gamma}{3 + 2\gamma + \hbar} \quad (5.83)$$

$$2: \quad \phi_2 = \Psi_2 + \frac{6\gamma}{3 + \gamma + 2\hbar} \Psi_1 + \frac{6(3 + \gamma + 2\gamma^2 + 2\hbar)}{(2 + \gamma + \hbar)(3 + \gamma + 2\hbar)}. \quad (5.84)$$

6. Summary and comments

The general theorem relating classical and quantum mechanics (section 2) is applied to the Calogero and Sutherland systems, typical integrable multi-particle dynamics associated with root systems and having long range interactions. The classical and quantum eigenfunctions for the elementary excitations are constructed explicitly (sections 4 and 5), and their relation to the eigenmodes of small oscillations (of the corresponding classical system) is worked out in full. In particular, we obtain new representations for the eigenvectors (of small oscillations) in terms of orthogonal polynomials of a discrete variable (the discrete variable being the zeros of well-known classical polynomials). It turns out that the quantum eigenfunctions are very closely related to the classical counterparts. As a special case, the quantum eigenfunction of the Sutherland system belonging to a minimal representation consists of a single Weyl orbit and it has exactly the same form as the classical one, that is the quantum corrections are absent. The next simplest case, belonging to the adjoint representations, is fully described by the number of roots and the highest exponents. As shown in many explicit examples, the classical and quantum eigenfunctions are fully described in terms of the roots, weights, exponents and characters, etc. We do believe that this is the case for any eigenfunctions of the Calogero and Sutherland systems. To demonstrate this assertion for any particular theory and universally for all the Calogero and Sutherland systems is a good challenge.

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